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Deforming a Lie algebra by means of a 2-form

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Abstract

We consider a vector space V over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, equipped with a skew symmetric bracket $[\cdot,\cdot] : V \times V \to V$ and a 2-form $\omega : V \times V \to \mathbb{K}$. A simple change of the Jacobi identity to the form $[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = \omega(B,C)A + \omega(A,B)C + \omega(C,A)B$ opens up new possibilities, which shed new light on the Bianchi classification of three-dimensional Lie algebras.

Keywords: Lie algebras; Bianchi classification; Deformation

1. Introduction

In Ref. [2] we considered a real vector space V of dimension n equipped with a Riemannian metric g and a symmetric 3-tensor Υ_{ijk} such that: (i) $\Upsilon_{ijk} = \Upsilon_{(ijk)}$, (ii) $\Upsilon_{ijj} = 0$ and (iii) $\Upsilon_{jki}\Upsilon_{lmi} + \Upsilon_{lji}\Upsilon_{kmi} + \Upsilon_{kli}\Upsilon_{jmi} = g_{jk}g_{lm} + g_{lj}g_{km} + g_{kl}g_{jm}$. Such a tensor defines a bilinear product $\{\cdot,\cdot\} : V \times V \to V$ given by

$$\{A,B\}_i = \Upsilon_{ijk}A_jB_k.$$

This product is symmetric:

$$\{A,B\} = \{B,A\} \tag{1.1}$$

due to property (i), and it satisfies a trilinear identity:

$$\{A,\{B,C\}\} + \{C,\{A,B\}\} + \{B,\{C,A\}\} = g(B,C)A + g(A,B)C + g(C,A)B, \tag{1.2}$$

due to property (iii). Restricting our attention to structures $(V, g, \{\cdot,\cdot\})$ associated with tensors Υ as above, we note that they are related to the isoparametric hypersurfaces in spheres [3,4]. Using Cartan's results [5] on isoparametric hypersurfaces we concluded in [6] that structures $(V, g, \{\cdot,\cdot\})$ exist only in dimensions 5, 8, 14 and 26.

A striking feature of property (1.2) is that it resembles very much the Jacobi identity satisfied by every Lie algebra. The main difference is that for a Lie algebra the bracket $\{\cdot,\cdot\}$ should be *anti*-symmetric and that the analog of (1.2) should have r.h.s. equal to *zero*.

Adapting properties (1.1) and (1.2) to the notion of a Lie algebra we are led to the following structure.

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Definition 1.1. A vector space V equipped with a bilinear bracket $[\cdot,\cdot] : V \times V \to V$ and a 2-form $\omega : V \times V \to \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ such that

$$[A, B] = -[B, A] \quad \text{and}$$
$$[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = \omega(B,C)A + \omega(A,B)C + \omega(C,A)B \tag{1.3}$$

is called an ω-deformed Lie algebra.

This definition obviously generalizes the notion of a Lie algebra and coincides with it when $\omega \equiv 0$. Note also that if the dimension of V is $\dim V = 2$, then $\omega(B,C)A + \omega(A,B)C + \omega(C,A)B \equiv 0$ for any 2-form ω and every $A, B, C \in V$. Thus in two dimensions it is impossible to ω-deform the Jacobi identity, and two-dimensional ω-deformed Lie algebras are just the Lie algebras equipped with a 2-form ω. This is no longer true if $\dim V \neq 2$. Indeed assuming that $\dim V \neq 2$ and that $\omega(B,C)A + \omega(A,B)C + \omega(C,A)B \equiv 0$ for all $A, B, C \in V$ we easily prove that $\omega \equiv 0$.

The aim of this note is to show that there exist ω-deformed Lie algebras in dimensions greater than 2 which are not just the Lie algebras.

2. Dimension 3

It follows that if $\dim V \leq 2$ then all the ω-deformed Lie algebras are just the Lie algebras. To show that in $\dim V = 3$ the situation is different we follow the procedure used in the Bianchi classification [1] of three-dimensional Lie algebras.

Let $\{e_i\}$, $i = 1, 2, 3$, be a basis of an ω-deformed three-dimensional Lie algebra. Then, due to the skew symmetry, we have $[e_i, e_j] = c^k_{ij} e_k$, $\omega(e_i, e_j) = \omega_{ij}$, where $c^k_{ij} = -c^k_{ji}$ and $\omega_{ij} = -\omega_{ji}$. Due to the ω-deformed Jacobi identity (1.3), we also have

$$c^m_{li} c^i_{jk} + c^m_{ki} c^i_{lj} + c^m_{ji} c^i_{kl} = \delta^m_l \omega_{jk} + \delta^m_k \omega_{lj} + \delta^m_j \omega_{kl},$$

which is equivalent to

$$c^m_{i[l} c^i_{jk]} + \delta^m_{[l} \omega_{jk]} = 0. \tag{2.1}$$

We now find all the orbits of the above defined pair of tensors (c^k_{ij}, ω_{ij}) under the action of the group $\mathbf{GL}(3, \mathbb{R})$.

We recall that in three dimensions, we have the totally skew symmetric Levi-Civita symbol ϵ_{ijk}, and its totally skew symmetric inverse ϵ^{ijk} such that $\epsilon_{ijk}\epsilon^{ilm} = \delta^l_j \delta^m_k - \delta^m_j \delta^l_k$. This can be used to rewrite the ω-deformed Jacobi identity (2.1). Indeed, since in three dimensions every totally skew symmetric 3-tensor is proportional to ϵ_{ijk}, the l.h.s. of (2.1) can be written as

$$t^m = 0 \quad \text{with } t^m = (c^m_{il} c^i_{jk} + \delta^m_l \omega_{jk})\epsilon^{ljk}.$$

In addition, we may use ϵ_{ijk} to write c^i_{jk} as

$$c^i_{jk} = n^{il}\epsilon_{jkl} - \delta^i_j a_k + \delta^i_k a_j, \tag{2.2}$$

where the symmetric matrix n^{il} is related to c^k_{ij} via

$$n^{il} = \frac{1}{2}(c^{il} + c^{li}), \quad \text{with } c^{il} = \frac{1}{2} c^i_{jk}\epsilon^{jkl}.$$

The vector a_m is related to c^k_{ij} via

$$a_m = \frac{1}{2}\epsilon_{mil} c^{il}.$$

Similarly, we write ω_{ij} as

$$\omega_{ij} = \epsilon_{ijk} b^k, \tag{2.3}$$

with

$$b^k = \frac{1}{2}\epsilon^{mik}\omega_{ik}.$$

Thus, in three dimensions the structural constants (c^k_{ij}, ω_{ij}) of the ω-deformed Lie algebra are uniquely determined via (2.2) and (2.3) by specifying a symmetric matrix n^{il} and two vectors a_m and b^k. In terms of the triple (n^{il}, a_m, b^k) the vector t^m is given by $t^m = 4n^{ml}a_l + 2b^m$, so that the ω-deformed Jacobi identity (2.1) is simply

$$b^i = -2n^{il}a_l. \tag{2.4}$$

Thus, given n^{il} and a_m, the vector b^m defining ω is totally determined. Now we use the action of the $\mathbf{GL}(3, \mathbb{R})$ group to bring n^{il} to the diagonal form (this is always possible since n^{il} is symmetric), so that

$$n^{il} = \mathrm{diag}(n^1, n^2, n^3).$$

It is obvious that without loss of generality we always can have

$$n^i = \pm 1, 0 \quad i = 1, 2, 3.$$

After achieving this we may still use an orthogonal transformation preserving the matrix n^{il} to bring the vector a_m to a form simpler than $a_m = (a_1, a_2, a_3)$. For example in the case $n^{il} = \mathrm{diag}(1, 1, 1)$ we may always achieve $a_m = (0, 0, a)$. Thus to represent a $\mathbf{GL}(3, \mathbb{R})$ orbit of (c^i_{jk}, ω_{ij}) it is enough to take n^{il} in the diagonal form with the diagonal elements equal to $\pm 1, 0$ and to take a_m in the simplest possible form obtainable by the action of $\mathbf{O}(n^{il})$. Finally we notice that the so specified choice of n^{il} is still preserved when the basis is scaled according to

$$e_1 \to \lambda_1 e_1, \qquad e_2 \to \lambda_2 e_2 \qquad e_3 \to \lambda_3 e_3, \tag{2.5}$$

with

$$(\lambda_1\lambda_2 - \lambda_3)n_3 = 0, \qquad (\lambda_3\lambda_1 - \lambda_2)n_2 = 0, \qquad (\lambda_2\lambda_3 - \lambda_1)n_1 = 0.$$

These transformations can be used to scale the vector a_m via

$$a_m \to (\lambda_1 a_1, \lambda_2 a_2, \lambda_3 a_3).$$

We are now in a position to give the full classification of three-dimensional ω-deformed Lie algebras. In all the types of the classification the commutation relations and the ω are given by

$$[e_1, e_2] = n^3 e_3 - a_2 e_1 + a_1 e_2, \qquad [e_3, e_1] = n^2 e_2 - a_1 e_3 + a_3 e_1, \qquad [e_2, e_3] = n^1 e_1 - a_3 e_2 + a_2 e_3$$
$$\omega(e_1, e_2) = -2n^3 a_3, \qquad \omega(e_3, e_1) = -2n^2 a_2, \qquad \omega(e_2, e_3) = -2n^1 a_1.$$

The classification splits into two main branches depending on whether a_m vanishes or not.

If $a_m = 0$, then $b^m = 0$. Thus $\omega = 0$ and all such structures correspond to the usual three-dimensional Lie algebras of the classical Bianchi types I, II, VI_0, VII_0, $VIII$ and IX (see the table below).

If $a_m \neq 0$ then, depending on the signature of n^{il}, vector a_m may be spacelike, timelike, null or degenerate. The orthogonal transformations that we use to normalize this vector preserve its type, so the classification splits according to the causal properties of a_m. If $n^2 = n^3 = 0$ or $n^1 = -n^2 = 1$, $n^3 = 0$, we may use transformations (2.5) to totally fix a_m. This leads to types V, IV, IV_T and VI_T, VI_S, VI_N below. In all other cases transformations (2.5) can be used to express a_m in terms of only one parameter $a > 0$ so that the different positive parameters a correspond to nonequivalent algebras. The resulting classification is summarized in the following table:

Bianchi type	n^1	n^2	n^3	a_m	b^m
I	0	0	0	(0, 0, 0)	(0, 0, 0)
II	1	0	0	(0, 0, 0)	(0, 0, 0)
VI_0	1	−1	0	(0, 0, 0)	(0, 0, 0)
VII_0	1	1	0	(0, 0, 0)	(0, 0, 0)
$VIII$	1	1	−1	(0, 0, 0)	(0, 0, 0)
IX	1	1	1	(0, 0, 0)	(0, 0, 0)
V	0	0	0	(0, 0, 1)	(0, 0, 0)
IV	1	0	0	(0, 0, 1)	(0, 0, 0)
IV_T	1	0	0	(1, 0, 0)	(−2, 0, 0)
VI_a	1	−1	0	$(0, 0, a > 0)$	(0, 0, 0)
VI_T	1	−1	0	(1, 0, 0)	(−2, 0, 0)
VI_S	1	−1	0	(0, 1, 0)	(0, 2, 0)
VI_N	1	−1	0	(1, 1, 0)	(−2, 2, 0)
VII_a	1	1	0	$(0, 0, a > 0)$	(0, 0, 0)
VII_T	1	1	0	(1, 0, 0)	(−2, 0, 0)
$VIII_a$	1	1	−1	$(0, 0, a > 0)$	$(0, 0, 2a)$
$VIII_{Ta}$	1	1	−1	$(a > 0, 0, 0)$	$(-2a, 0, 0)$
$VIII_{Na}$	1	1	−1	$(a > 0, 0, a)$	$(-2a, 0, 2a)$
IX_a	1	1	1	$(0, 0, a > 0)$	$(0, 0, -2a)$

Note that all the types which have $b^m = 0$ are just the usual three-dimensional Lie algebras. Apart from the already mentioned types with $a^m = 0$ these are the classical Bianchi types: V, IV, VI_a (with $VI_1 = III$) and VII_a. With the exception of the types I and V all the Bianchi types admit ω deformation. It is interesting to note that types $VIII$ and IX, which in the Lie algebra setting do not admit $a_m \neq 0$ deformation, admit a one-parameter ω-deformation.

We have the following theorem.

Theorem 2.1. *All the three-dimensional ω-deformed Lie algebras with $\omega \neq 0$ are given in the following table:*

Bianchi type	n^1	n^2	n^3	(a_1, a_2, a_3)	(b^1, b^2, b^3)
IV_T	1	0	0	(1, 0, 0)	(−2, 0, 0)
VI_T	1	−1	0	(1, 0, 0)	(−2, 0, 0)
VI_S	1	−1	0	(0, 1, 0)	(0, 2, 0)
VI_N	1	−1	0	(1, 1, 0)	(−2, 2, 0)
VII_T	1	1	0	(1, 0, 0)	(−2, 0, 0)
$VIII_a$	1	1	−1	$(0, 0, a > 0)$	$(0, 0, 2a)$
$VIII_{Ta}$	1	1	−1	$(a > 0, 0, 0)$	$(-2a, 0, 0)$
$VIII_{Na}$	1	1	−1	$(a > 0, 0, a)$	$(-2a, 0, 2a)$
IX_a	1	1	1	$(0, 0, a > 0)$	$(0, 0, -2a)$

They satisfy the commutation relations

$$[e_1, e_2] = n^3 e_3 - a_2 e_1 + a_1 e_2, \qquad [e_3, e_1] = n^2 e_2 - a_1 e_3 + a_3 e_1, \qquad [e_2, e_3] = n^1 e_1 - a_3 e_2 + a_2 e_3$$
$$\omega(e_1, e_2) = -2n^3 a_3, \qquad \omega(e_3, e_1) = -2n^2 a_2, \qquad \omega(e_2, e_3) = -2n^1 a_1$$

with the real parameters $(n^1, n^2, n^3, a_1, a_2, a_3)$ specified in the table. Algebras corresponding to different $(n^1, n^2, n^3, a_1, a_2, a_3)$ are nonequivalent.

Finally we show that any ω-deformed Lie algebra must have quite nontrivial structure constants. Indeed, in any dimension $\dim V = n > 2$ the structure constants of an ω-deformed Lie algebra, which are defined by $[e_i, e_j] = c^k_{ij} e_k$, may be decomposed as follows:

$$c^i_{jk} = \alpha^i_{jk} + a_k \delta^i_j - a_j \delta^i_k,$$

where

$$\alpha^i_{ik} = 0, \qquad a_k = \frac{1}{n-1} c^i_{ik}.$$

Then a simple calculation using the ω-deformed Jacobi identity (1.3) shows that

$$\omega(e_j, e_k) = \frac{n-1}{n-2} a_i \alpha^i_{jk}.$$

This shows that nonvanishing ω is only possible if both a_i and α^i_{jk} are nonvanishing.

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